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Two-Time-Scale Relaxation Towards Thermal Equilibrium of the Enigmatic Piston

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We investigate the evolution of a system composed of N non-interacting point particles of mass m in a container divided into two chambers by a movable adiabatic piston of mass $M \gg m$. Using a two-time-scale perturbation approach in terms of the small parameter $\alpha = 2m/(M+m)$, we show that the evolution towards thermal equilibrium proceeds in two stages. The first stage is a fast, deterministic, adiabatic relaxation towards mechanical equilibrium. The second stage, which takes place at times $\mathcal{O}(M)$, is a slow fluctuation-driven, diathermic relaxation towards thermal equilibrium. A very simple equation is derived which shows that in the second stage, the position of the piston is given by $X_M(t) = L[1/2 - \xi(\alpha t)]$ where the function ξ is independent of M. Numerical simulations support the assumptions underlying our analytical derivations and illustrate the large mass range in which the picture holds.

KEY WORDS: Liouville equation; adiabatic; mechanical equilibrium; thermal equilibrium; perturbation.

1. INTRODUCTION

The "adiabatic" piston problem is a well-known controversial example of thermodynamics.[1] An isolated cylinder contains two identical fluids which initially are in different equilibrium states and which are separated by an adiabatic fixed piston. The whole system remains therefore in equilibrium. The problem is then to predict the final state to which the system will evolve when the constraint fixing the piston is released. Although it is a very old problem since it was discussed to measure experimentally the ratio

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c_p/c_v of gases already before 1940,[2, 3] it still remains a controversial question since it shows that the two laws of equilibrium thermodynamics are not sufficient to predict the final state.

Recently, especially after the work on the second law of thermodynamics by Lieb and Ynvagson[4] and the talk of Lieb at the StatPhys 20 meeting in 1998,[5] this problem has attracted renewed interest. It was first realized that it is a standard example where one is forced to apply non-equilibrium thermodynamics since the final state may depend on the values of the friction coefficients.[6] Then a very simple model was considered to investigate the evolution from a microscopical point of view.[7] This system consists of N non-interacting particles in a cylinder of length L and cross-section A. It is divided into two compartments containing respectively N^- and N^+ particles ($N = N^- + N^+$) by an adiabatic (i.e., no internal degrees of freedom) piston of mass M. The dynamics is defined by the condition that the piston is constrained to move without friction along the x-axis and the particles make purely elastic collisions on the boundaries and on the piston. Without loss of generality, we can assume that all the particles have velocities parallel to the x-axis and thus we are led formally to a one-dimensional problem (except for normalization). Therefore, given that the velocities of a particle and the piston are v and V before they collide, then after the collision the velocities will be v' and V', with:

$$v' = 2V - v + \alpha(v - V) \qquad V' = V + \alpha(v - V) \qquad \text{where} \quad \alpha = \frac{2m}{M+m} \tag{1}$$

For physical situations where $m \ll M$, this model was investigated in ref. 8 using Boltzmann equation and a perturbation expansion of the velocity distribution function of the piston $\Phi_\epsilon(V, t)$ in powers of $\epsilon = \sqrt{m/M}$. For an infinite cylinder ($L = \infty$) and equal pressures on both sides of the piston ($p^- = p^+$), it was shown that the stationary solution of the Boltzmann equation gives a constant velocity for the piston $V_{st} = (1/4M)\sqrt{2\pi k_B m} \times (\sqrt{T^+} - \sqrt{T^-}) + \mathcal{O}(m/M)$ towards the high temperature domain. It was thus concluded that stochastic motion together with space asymmetry (temperature difference) implies a macroscopic motion. Using qualitative arguments and numerical simulations, the case of a finite cylinder was investigated in ref. 9. It was thus realized that the evolution takes place in two or three stages. In a first stage, the evolution appears to be deterministic and adiabatic; this first stage proceeds until mechanical equilibrium is reached ($p^- = p^+$ but $T^- \neq T^+$). In the second stage which takes place on much longer time scale, the simulations showed that the motion of the piston is stochastic and proceeds with exchange of heat through the piston

until thermal equilibrium is reached where $T^- = T^+$, i.e., the piston which was adiabatic when fixed and during the first stage becomes heat-conducting under fluctuations. This explains in particular the results mentioned above for the infinite cylinder with $p^- = p^+$. In the third stage, under the stochastic motion of the piston, the velocity distribution functions of the fluids tend to Maxwellian distributions.

Recently we have studied the adiabatic piston problem in the thermodynamic limit for the piston, i.e., by considering the limit where L is fixed but the area A of the cylinder, the mass M of the piston and the number $N^\pm$ of fluid particles tend to infinity with M/A and $R^\pm = mN^\pm/M$ fixed.[10] Starting from Liouville equation, it was shown that in this thermodynamic limit, the motion of the piston is adiabatic and deterministic, i.e., $\langle V^n \rangle_t = \langle V \rangle_t^n$. Introducing at this point simplifying assumptions (see Assumptions 2 and 3 later), we obtained a system of autonomous equations from which we concluded that the system evolves towards a state of mechanical equilibrium where the pressures are equal but the temperatures different. Furthermore, numerical simulations were conducted which showed that the motion depends strongly on $R^\pm$ for $R^\pm < 1$ but tends to be independent of $R^\pm$ for $R^\pm > 10$. For $R^\pm < 1$, the evolution is very weakly damped and the period of oscillations, which depends on $R^\pm$, coincide with the period computed with our equations, and with the period obtained from thermodynamics assuming adiabatic oscillations.[2,3] On the other hand for $R^\pm > 10$, the motion is strongly damped and is independent of $R^\pm$. An equation was thus conjectured to describe the evolution for $R^\pm$ large enough. It should be noticed that these two types of regimes, i.e., weak vs. strong damping, have been observed experimentally,[3] as well as in the previous numerical simulations for the simple piston where the evolution is necessarily adiabatic.[11] Let us also stress that in this thermodynamic limit, we could deduce from Liouville equation the factorization property of the joint velocity distribution for the piston and a particle at the piston surface.

In ref. 9 were presented the first simulations for this model and a qualitative (non-rigourous) discussion was given to explain the numerical results. In particular, for finite systems, this first attempt was heuristic and the condition introduced to take into account the pressure difference between the left and right compartments was not at all clear. In ref. 10 a rigourous analysis (i.e., under well defined assumptions) of the adiabatic evolution was given in the thermodynamic limit discussed above. In this thermodynamic limit a factorisation property was obtained and it was shown that the system evolves toward mechanical equilibrium. In the present article, we investigate the motion of the piston with M finite but $M \gg m$. Introducing now the factorization property as an assumption (see Assumption 1 later) which we assume to be valid to first order in α, and

using a two-time-scale perturbation approach, we show that the evolution of the piston proceeds in two stages with totally different properties and time scales. In the first stage, characterized by a time scale $t_1 = L\sqrt{Nm/E_0}$ (average time for a particule to collide twice with the piston) where E_0 is the initial energy, the pressure and temperature differences between the two gases are of order zero in m/M. In this stage the evolution is similar to the evolution obtained in the previous thermodynamic limit:[10] it is adiabatic, deterministic, independent of M for M sufficiently large, either weakly or strongly damped, and proceeds until mechanical equilibrium is reached (up to corrections in m/M).

In the second stage, characterized by a time scale $t_2 = Mt_1/m$, the evolution is strongly dependent on M. In fact, we shall derive the following scaling property:

Main Result. Introducing the scaled time variable $\tau = \alpha t$, the evolution of the piston of mass $M \gg m$ is given by:

$$X_M(\tau) = L(\tfrac{1}{2} - \xi(\tau)) \tag{2}$$

where

$$\frac{d\xi}{d\tau} = -\frac{1}{3}\frac{1}{L}\sqrt{\frac{E_0}{N}}\sqrt{\frac{8}{m\pi}}\left[\sqrt{\frac{N}{N^+}(1+2\xi)} - \sqrt{\frac{N}{N^-}(1-2\xi)}\right] \tag{3}$$

Moreover from the knowledge of $\xi(\tau)$, one obtains the temperatures and pressures on both sides of the piston, as well as the temperature of the piston, to lowest significant order in α. In this second stage, the fluctuation-driven evolution is stochastic and proceeds with heat transfer across the piston towards a state of "thermal" equilibrium where $T^- = T^+$. Investigation to higher orders in α would be necessary to conclude that the fluid velocity distributions will ultimately tend to Maxwellian distributions. Let us also stress that the initial state for the second stage is the final stage for the first stage (matching condition).

The same microscopical model was analyzed by Lebowitz, Piasecki, and Sinai using a different limiting procedure.[13] In their work, they have considered the container to be a cube of size L and they have taken the limit $L \to \infty$ with $N^- = N^+ \sim L^3$ and $M \sim L^2$. Using heuristical arguments, they derived autonomous coupled equations describing the motion of the piston and the fluid for large L. More recently, exact results were obtained by Chernov, Lebowitz, and Sinai using this limiting procedure $L \to \infty$.[14] They were able to prove rigorously that for a time interval sufficiently short so that only the first and second recollisions of a particle on the piston can occur, the random functions describing the position and the

velocity of the piston, expressed in terms of scaled variables ($\tau = t/L$, $Y = X/M$), converge in probability to some deterministic functions. However the time interval to which their results hold is inversely proportional to an arbitrary introduced cutoff and is zero for Maxwellian distributions. On the other hand, with the natural cutoff which appears in numerical simulations, this time interval is of the order of the time needed for the first oscillation to occur.

To conclude this introduction, we should mention that the approach to "thermal" equilibrium was investigated in ref. 15 using an expansion of the master equation for the piston velocity distribution function in powers of $\sqrt{\alpha}$, as well as numerical simulations, for a gas of hard rods.

We shoud insist on the fact that our model is formally a one-dimensional system and that the particles do not interact. However it appears that similar results will also hold for two dimensional systems of hard-disks. Indeed an interesting discussion was presented in ref. 16, where the power spectrum and the time correlation function of the piston were calculated at equilibrium for a two-dimensional system of hard disks by molecular dynamics simulations. From this analysis, one can also deduce a two-time-scale relaxation towards equilibrium. The problem of approach to equilibrium and the question of heat transfer for this 2-dimensional system of hard disks was analyzed in refs. 17 and 18 using molecular dynamics simulations and a linear Langevin equation (see also ref. 11 for a 2-dimensional system of hard-disks). It is however possible that the evolution toward Maxwellian distribution is faster in higher dimension and for interacting molecules, but we have not studied this question.

Let us finally mention another problem which was investigated in 1963.[12] It presents similarity with our model (if the cylinder has infinite length) in the sense that there is a stochastic motion coupled to a space asymmetry. However the conditions are very different since in the present work the two gases are identical but the thermodynamical conditions are different, while in ref. 12 the two gases are different but the thermodynamical conditions are the same. Both ref. 12 and the following analysis are based on perturbation with respect to the small parameter m/M, but the methods are entirely different.

2. EQUATIONS FOR THE MOMENTS $\langle V^n \rangle$ OF THE PISTON VELOCITY

In the spirit of BBGKY hierarchy, we shall characterize the velocity distribution $\Phi(V, t)$ through its moments:

$$\bar{V}(t) = \langle V \rangle(t) = \int_{-\infty}^{\infty} V \Phi(V, t)\, dV \tag{4}$$

and

$$\langle V^n \rangle(t) = \int_{-\infty}^{\infty} V^n \Phi(V, t)\, dV \tag{5}$$

Introducing the notation:

$$\tilde{F}_k(V, \rho^{\pm}_{\text{surf}}) = \left[\int_V^{\infty} (v-V)^k \rho^{-}_{\text{surf}}(v, V, t)\, dv - \int_{-\infty}^{V} (v-V)^k \rho^{+}_{\text{surf}}(v, V, t)\, dv \right] \tag{6}$$

where $k \geqslant 0$ and $\rho^{\pm}_{\text{surf}}(v, V, t)$ is the two-point correlation function for one molecule on the left $(-)$, resp. on the right $(+)$, and the piston, then by integration of Liouville equation over all variables except the piston velocity V, we have obtained the following evolution equation for Φ:[10]

$$\frac{1}{\gamma} \partial_t \Phi(V, t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \alpha^k}{(k+1)!} \frac{\partial^{k+1} \tilde{F}_{k+2}}{\partial V^{k+1}} (V, \rho^{\pm}_{\text{surf}}) \tag{7}$$

where $\gamma = A\alpha = 2mA/(M+m)$ and with $\partial_t \Phi(V, t)$ defined in the framework of generalized functions. Hence, *by definition*, integration over V commute with ∂_t and with the infinite summation in any average with respect to Φ:

$$\begin{aligned} \frac{1}{\gamma} \frac{d\langle h(V) \rangle}{dt} &= \int_{-\infty}^{\infty} h(V)\, \partial_t \Phi(V, t)\, dV \\ &= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} h(V) \frac{(-1)^{k+1} \alpha^k}{(k+1)!} \frac{\partial^{k+1} \tilde{F}_{k+2}}{\partial V^{k+1}} (V, t)\, dV \end{aligned} \tag{8}$$

In particular, we have:

$$\frac{1}{\gamma} \frac{d\langle V^n \rangle}{dt} = \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} V^n \frac{(-1)^{k+1} \alpha^k}{(k+1)!} \frac{\partial^{k+1} \tilde{F}_{k+2}}{\partial V^{k+1}} (V, \rho^{\pm}_{\text{surf}})\, dV \tag{9}$$

Under the hypothesis that $V \to \rho^{\pm}_{\text{surf}}(v, V, t)$ decreases faster than any power of V, uniformly in v, we may integrate by parts, which yields:

$$\begin{aligned} \frac{1}{\gamma} \frac{d\langle V^n \rangle}{dt} = &\sum_{k=0}^{n-1} \int_{-\infty}^{\infty} \frac{n!\, \alpha^k}{(k+1)!} \frac{V^{n-k-1}}{(n-k-1)!} \tilde{F}_{k+2}(V, \rho^{\pm}_{\text{surf}})\, dV \\ &+ \sum_{k=n}^{\infty} \int_{-\infty}^{\infty} \frac{n!\, (-1)^{k+1+n} \alpha^k}{(k+1)!} \frac{\partial^{k+1} \tilde{F}_{k+2}}{\partial V^{k+1}} (V, \rho^{\pm}_{\text{surf}})\, dV \end{aligned} \tag{10}$$

All terms with $k \geqslant n$, appearing as pure derivatives with respect to V, vanish after integration on the whole V-axis, due to the fast decrease of $V \to \rho^{\pm}_{\text{surf}}(v, V, t)$ at infinity. Hence the equation simplifies into:

$$\frac{1}{\gamma}\frac{d\langle V^n\rangle}{dt} = \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} \frac{n!\,\alpha^k}{(k+1)!}\frac{V^{n-k-1}}{(n-k-1)!}\tilde{F}_{k+2}(V, \rho^{\pm}_{\text{surf}})\,dV \tag{11}$$

where $\tilde{F}_k$ depends functionally on $\rho^{\pm}_{\text{surf}}(v, V, t)$.

3. FACTORIZATION PROPERTY

In ref. 10, we have shown that in the limit $\alpha = 0$, the two-point correlation functions $\rho^{\pm}_{\text{surf}}(v, V, t)$ have the factorization property. Although we can not prove it, we expect that in the limit $\alpha \to 0$ which we now consider, and for initial conditions such that the evolution is smooth, this property will still be valid to first order $\mathcal{O}(\alpha)$ where the leading behavior is $\Phi(V, t) = \delta(V - \bar{V}(t))$. Therefore in the following perturbation approach, we shall consider that the following assumption holds to first order in α.

Assumption 1 (Factorization Property). Before a collision the two-point correlation functions have the following factorization property at first order in α:

$$\rho^{\pm}_{\text{surf}}(v, V, t) = \rho^{\pm}_{\text{surf}}(v, t)\,\Phi(V, t) \tag{12}$$

This assumption is in fact of the same nature as the molecular chaos assumption introduced in kinetic theory. Under this factorization property, we have $\tilde{F}_k = F_k\Phi$ where:

$$\begin{aligned} F_k(V, \rho^{\pm}_{\text{surf}}) &= F^{-}_k(V, \rho^{-}_{\text{surf}}) - F^{+}_k(V, \rho^{+}_{\text{surf}}) \\ &= \int_V^{\infty} (v-V)^k\,\rho^{-}_{\text{surf}}(v, t)\,dv - \int_{-\infty}^{V} (v-V)^k\,\rho^{+}_{\text{surf}}(v, t)\,dv \end{aligned} \tag{13}$$

and the evolution of the moments $\langle V^n\rangle$ of the piston velocity satisfy the equation:

$$\frac{1}{\gamma}\frac{d\langle V^n\rangle}{dt} = \sum_{k=0}^{n-1} \int_{-\infty}^{\infty} \frac{n!\,\alpha^k}{(k+1)!}\frac{V^{n-k-1}}{(n-k-1)!}F_{k+2}(V, \rho^{\pm}_{\text{surf}})\,\Phi(V, t)\,dV \tag{14}$$

Similarly, under the factorization assumption, the distributions $\rho^{\pm}(x, v, t)$ of the fluid molecules satisfy the Boltzmann equations with boundaries:[10]

$$\begin{aligned}(\partial_t + v\partial_x)\,\rho^-(x,v,t) &= \delta(x)\,v\rho^-(0,v,t)\\ &\quad +\delta(x-X(t))[V(t)-v][\theta(V(t)-v)\,\rho^-(X(t),v',t)\\ &\quad +\theta(v-V(t))\,\rho^-(X(t),v,t)] \qquad (15)\\ (\partial_t + v\partial_x)\,\rho^+(x,v,t) &= -\delta(x-L)\,v\rho^+(L,v,t)\\ &\quad -\delta(x-X(t))[V(t)-v][\theta(v-V(t))\,\rho^+(X(t),v',t)\\ &\quad +\theta(V(t)-v)\,\rho^+(X(t),v,t)] \qquad (16)\end{aligned}$$

where $v' = 2V - v + \alpha(v-V)$ is the velocity of the molecules after their collision onto the piston. Note that $\rho^{\pm}(X(t), v, t)$ is what we denote $\rho^{\pm}_{\text{surf}}(v, t)$.

It is straighforward to check that:

$$\frac{dF^{\pm}_k}{dV}(V, \rho^{\pm}_{\text{surf}}) = -kF^{\pm}_{k-1}(V, \rho^{\pm}_{\text{surf}}) \qquad (\text{if} \quad k \geqslant 1) \qquad (17)$$

which leads to define $F^{\pm}_k$ for $k < 0$ by:

$$\left(\frac{d}{dV}\right)^r F^{\pm}_0 \doteq F^{\pm}_{-r} = \pm\left(\frac{d}{dV}\right)^{r-1} \rho^{\pm}_{\text{surf}}(V, t) \qquad (r \geqslant 1) \qquad (18)$$

We define the densities $\rho^{\pm}_{\text{surf}}$ (not to be confused with the distributions $\rho^{\pm}_{\text{surf}}(v)$), the temperatures $T^{\pm}_{\text{surf}}$ and the pressures $p^{\pm}_{\text{surf}}$ at the surface of the piston, on each side:

$$\rho^-_{\text{surf}} \doteq 2\int_0^{\infty} \rho^-_{\text{surf}}(v, t)\,dv = 2F^-_0(V = 0, \rho^-_{\text{surf}}) \qquad (19)$$

$$\rho^+_{\text{surf}} \doteq 2\int_{-\infty}^{0} \rho^+_{\text{surf}}(v, t)\,dv = 2F^+_0(V = 0, \rho^+_{\text{surf}}) \qquad (20)$$

$$p^{\pm}_{\text{surf}} \doteq 2mF^{\pm}_2(V = 0) \doteq \rho^{\pm}_{\text{surf}} k_{\text{B}} T^{\pm}_{\text{surf}} \qquad (21)$$

It is obvious from the definition that $F^-_2(V)$ and $-F^+_2(V)$ are decreasing functions of V, which allows to express $F^{\pm}_2(V)$ and $F^{\pm}_0(V)$ in the following form:

$$2mF_2^{\pm}(V) \doteq p_{\text{surf}}^{\pm} \pm \left(\frac{M+m}{A}\right) \lambda^{\pm}(V)\, V \tag{22}$$

$$2F_0^{\pm}(V) \doteq \rho_{\text{surf}}^{\pm} \pm \left(\frac{M+m}{A}\right)\left(\frac{\tilde{\lambda}^{\pm}(V)}{k_B T_{\text{surf}}^{\pm}}\right) V \tag{23}$$

and $\lambda^{\pm}(V)$ have the physical meaning of friction coefficients. Denoting $\lambda(V) = \lambda^{-}(V) + \lambda^{+}(V)$ yields:

$$2mF_2(V) = (p^{-} - p^{+}) - \left(\frac{M+m}{A}\right) \lambda(V)\, V \tag{24}$$

From Eq. (17), we have also $2mF_2^{\pm}(V) = p^{\pm} - 4mF_1^{\pm}(0)\, V + \mathcal{O}(V^2)$, so that, denoting simply $\lambda^{\pm} = \lambda^{\pm}(V=0)$:

$$F_1^{\pm}(V=0) = \mp \frac{1}{4m}\left(\frac{M+m}{A}\right) \lambda^{\pm} \tag{25}$$

4. IRREDUCIBLE MOMENTS

We denote $F_n^{(r)}$ the rth derivative of F_n (with respect to V); from Eq. (17), these $F_n^{(r)}$ are functional of $\rho_{\text{surf}}^{\pm}(v, t)$, proportional to $F_{n-r}(V, \rho_{\text{surf}}^{\pm})$. We thus have:

$$F_n(V, \rho_{\text{surf}}^{\pm}) = \sum_{r=0}^{\infty} \frac{1}{r!} F_n^r(\bar{V}, \rho_{\text{surf}}^{\pm})(V - \bar{V})^r \tag{26}$$

where:

$$(F_n^{\pm})^{(r)} = (-1)^r \frac{n!}{(n-r)!} F_{n-r}^{\pm} \qquad \text{for} \quad r \leqslant n \tag{27}$$

$$(F_n^{\pm})^{(r)} = \pm (-1)^n\, n! \left(\frac{d}{dV}\right)^{r-n-1} \rho_{\text{surf}}^{\pm}(V, t) \qquad \text{for} \quad r \geqslant n+1 \tag{28}$$

Plugging the expansion Eq. (26) in Eq. (14) yields for $n = 1$:

$$\frac{1}{\gamma}\frac{d\bar{V}}{dt} = F_2(\bar{V}, t) + \sum_{r \geqslant 2} \frac{\Delta_r}{r!} F_2^{(r)}(\bar{V}, t) \tag{29}$$

where:

$$\begin{gathered} F_2^{(0)} = F_2, \qquad F_2^{(1)} = -2F_1, \qquad F_2^{(2)} = 2F_0, \\ F_2^{(3)} = 2F_{-1} = -2[\rho_{\text{surf}}^{-}(V, t) + \rho_{\text{surf}}^{+}(V, t)] \end{gathered} \tag{30}$$

and Δ_r are the irreducible moments:

$$\Delta_r \equiv \int_{-\infty}^{\infty} (V-\bar{V}(t))^r \, \Phi(V,t)\, dV = \sum_{q=0}^{r} (-1)^q \frac{r!}{q!(r-q)!} \bar{V}(t)^q \langle V^{r-q} \rangle \tag{31}$$

with

$$\bar{V}(t) = \int_{-\infty}^{\infty} V \Phi(V,t)\, dV \tag{32}$$

the average velocity of the piston. Let us note that

$$\Delta_0 = 1 \tag{33}$$

$$\Delta_1 = 0 \tag{34}$$

$$\Delta_2 = \langle V^2 \rangle - \bar{V}^2(t) \tag{35}$$

We can invert Eq. (31) to express the moments $\langle V^s \rangle$ as functions of the irreducible moments (cumulant expansion):

$$\langle V^s \rangle = \bar{V}^s + \sum_{q=2}^{s} \frac{s!}{q!\,(s-q)!} \bar{V}^{s-q} \Delta_q \tag{36}$$

Finally replacing the function $F_{k+2}(V, \rho^{\pm}_{\text{surf}})$ by its expansion in powers of $(V-\bar{V}(t))$ around the average velocity $\bar{V}(t)$ and using Eq. (36) lead to the following evolution equations for the irreducible moments ($s \geqslant 2$):

$$\frac{1}{\gamma}\frac{d\Delta_s}{dt} = -s\Delta_{s-1}\left[\sum_{r\geqslant 2} \frac{1}{r!}\Delta_r F_2^{(r)}\right] - 2s\left[\sum_{n\geqslant 0} \frac{1}{(n+1)!}\Delta_{s+n} F_1^{(n)}\right] + \alpha\left[\sum_{k=0}^{s-2} \alpha^k \frac{s!}{(k+2)!\,(s-2-k)!}\left[\sum_{n\geqslant 0} \frac{1}{n!}\Delta_{s-2-k+n} F_{3+k}^{(n)}\right]\right] \tag{37}$$

where we recall that F_k are functions of $\bar{V}$ and functional of $\rho^{\pm}(\cdot, t)$.

For $s = 2$, it comes:

$$\frac{1}{\gamma}\frac{d\Delta_2}{dt} = -4 \sum_{n\geqslant 0} \frac{\Delta_{n+2}}{(n+1)!} F_1^{(n)} + \alpha \sum_{n\geqslant 0} \frac{1}{n!}\Delta_n F_3^{(n)} \tag{38}$$

Using the fact that $\Delta_1 = 0$, $F_1^{(n)} = -\frac{1}{2} F_2^{(n-1)}$, and $F_3^{(r)} = -3F_2^{(r-1)}$ if $r \geqslant 1$, we may rewrite this equation:

$$\frac{1}{\gamma}\frac{d\Delta_2}{dt} = \sum_{r\geqslant 2} \frac{\Delta_r}{r!}(2r-3\alpha)\, F_2^{(r-1)} + \alpha F_3 \tag{39}$$

5. EVOLUTION AT FIRST ORDER IN α

A qualitative analysis of Eq. (37) shows that:

$$\Delta_s \sim \alpha^{\left[\frac{(s+1)}{2}\right]} \tag{40}$$

where $[(s+1)/2]$ is the integral part of $(s+1)/2$. Since we want to restrict our study to first order in α, assuming that Eq. (40) is valid, we only have to consider the evolution of $\bar{V}$ and Δ_2, Eqs. (29) and (39), which we supplement with the equations for the energies $\langle E^{\pm}\rangle$ of the gas in the left and right compartments, restricted similarly to first order in α. We thus obtain to first order in α the following set of coupled *deterministic* equations:

$$\frac{1}{\gamma}\frac{d\bar{V}}{dt} = F_2 + \Delta_2 F_0 \tag{41}$$

$$\frac{1}{\gamma}\frac{d\Delta_2}{dt} = -4\Delta_2 F_1 + \alpha F_3 \tag{42}$$

$$\frac{1}{\gamma}\frac{d\langle E^-\rangle}{dt} = -M\bar{V}[F_2^- + \Delta_2 F_0^-] + \frac{M}{2}[4\Delta_2 F_1^- - \alpha F_3^-] \tag{43}$$

$$\frac{1}{\gamma}\frac{d\langle E^+\rangle}{dt} = M\bar{V}[F_2^+ + \Delta_2 F_0^+] - \frac{M}{2}[4\Delta_2 F_1^+ - \alpha F_3^+] \tag{44}$$

and we recall that all the functions F are functions of $\bar{V}(t)$ and functionals of $\rho^{\pm}_{\text{surf}}(v, t)$. We should note that the set of Eqs. (41)–(44) have a constant of motion, i.e.,

$$\langle E^-\rangle + \langle E^+\rangle + \tfrac{1}{2}M(\bar{V}^2 + \Delta_2) = E_0 = \text{const.} \tag{45}$$

which reflects the conservation of energy at the microscopic level.

Remark 1. The evolution is described by Eqs. (41) and (42) together with Boltzmann equations for the fluid Eqs. (15) and (16). In the next section, we shall introduce the "average assumption" which will then enable us to ignore the equation for the fluids and to replace them by Eqs. (43) and (44).

Remark 2. Recalling that $\Delta_2(t=0) = 0$, it is straighforward using Gronwall lemma to check the consistency of the scaling hypothesis, i.e., $\Delta_2 = \mathcal{O}(\alpha)$, from the above evolution equation (42). Indeed, F_3 remains of

order $\mathcal{O}(1)$ (upper bound denoted $\sup(F_3)$) and F_1 is strictly positive since $F_1 = 0$ would imply that:

$$\begin{cases} \rho^-_{\text{surf}}(v, t) = 0 & \forall v \geqslant \bar{V}(t) \\ \rho^+_{\text{surf}}(v, t) = 0 & \forall v \leqslant \bar{V}(t) \end{cases} \tag{46}$$

which is precluded by the relations:

$$\begin{cases} \forall v \geqslant \bar{V}(t) & (1-\alpha)\ \rho^-_{\text{surf}}(v, t) = \rho^-_{\text{surf}}(2\bar{V}_t - v + \alpha(v - \bar{V}_t), t) \\ \forall v \leqslant \bar{V}(t) & (1-\alpha)\ \rho^+_{\text{surf}}(v, t) = \rho^+_{\text{surf}}(2\bar{V}_t - v + \alpha(v - \bar{V}_t), t) \end{cases} \tag{47}$$

Therefore Δ_2 is bounded by $\alpha \sup(F_3)/\inf(F_1)$ i.e., $\Delta_2 = \mathcal{O}(\alpha)$.

Remark 3. In our first paper,[10] we have considered the case $\alpha = 0$, i.e., the thermodynamic limit for the piston, and discussed the evolution described by $\Delta_2 = 0$ and:

$$\frac{1}{\gamma}\frac{dV}{dt} = F_2(V) \tag{48}$$

$$\frac{1}{\gamma}\frac{d\langle E^- \rangle}{dt} = -MVF_2^-(V) \tag{49}$$

$$\frac{1}{\gamma}\frac{d\langle E^+ \rangle}{dt} = MVF_2^+(V) \tag{50}$$

6. ADIABATIC EVOLUTION FOR SHORT TIME

We want to investigate the evolution of the piston under the initial condition where $X(t = 0) = X_0$, $V(t = 0) = 0$ and the fluids on both sides of the piston are in equilibrium, described by Maxwellian distribution of velocity, with temperatures $T_0^{\pm}$. The initial conditions are such that $|T_0^+ - T_0^-| = \mathcal{O}(1)$ and $|p_0^+ - p_0^-| = \mathcal{O}(1)$. Since $\alpha \ll 1$, the perturbation approach shows that in a first stage, we can restrict our analysis to order zero in α and thus we recover the results of our previous paper (where $\alpha = 0$), except that they will now be valid only for finite time. In this first stage, the motion of the piston is a *deterministic* (no velocity fluctuations, $\Delta_2 = 0$) *adiabatic* (no heat transfer between the compartments) evolution towards mechanical equilibrium. This first stage ends when $p^- - p^+ = \mathcal{O}(\alpha)$ (but $T^- - T^+$ is still $\mathcal{O}(1)$). At this point, the term F_2 which appears in the evolution of $\bar{V}$, Eq. (41), becomes of order α and the first order terms of the perturbation approach must now be taken into account. Let us note that if

we introduce the "average assumption" (see Assumption 2 later), the state at the end of this first stage is given by the final state derived in ref. 10 for $\alpha = 0$:

$$p_{\mathrm{ad}}^{\pm} = p_0 + \mathcal{O}(\alpha) \tag{51}$$

$$T_{\mathrm{ad}}^{-} = \left(\frac{N}{N^-}\right) T_0 \frac{X_{\mathrm{ad}}}{L} + \mathcal{O}(\alpha) \tag{52}$$

$$T_{\mathrm{ad}}^{+} = \left(\frac{N}{N^+}\right) T_0 \left(1 - \frac{X_{\mathrm{ad}}}{L}\right) + \mathcal{O}(\alpha) \tag{53}$$

where:

$$k_{\mathrm{B}} T_0 = \left(\frac{AL}{N}\right) p_0 = \frac{2E_0}{N}, \qquad N = N^- + N^+, \tag{54}$$

and X_{ad} is the solution of

$$\sqrt{\left(\frac{N}{N^-}\right) X_{\mathrm{ad}}^3} - \sqrt{\left(\frac{N}{N^+}\right) (L - X_{\mathrm{ad}})^3} = \sqrt{\frac{L}{T_0}}\, C \tag{55}$$

where the constant C is related to the initial conditions according to

$$C = \sqrt{T^-(0)}\, X_0 - \sqrt{T^+(0)} (L - X_0) \tag{56}$$

7. SLOW RELAXATION TOWARDS THERMAL EQUILIBRIUM

The perturbation approach developped here (at order 1 in α) intends to reach the long-time behavior of the piston motion in the case where $M \gg m$. We claim that the fluctuations of the piston velocity V enter the scene only in a second stage, of time scale $\mathcal{O}(1/\alpha)$, once the pressure difference has become of order $\mathcal{O}(\alpha)$. In order to prove this assertion, we have to develop a boundary-layer-type perturbation approach;[21] indeed, at the end of the fast relaxation towards mechanical equilibrium, the standard perturbation approach (with time variable t) becomes singular and fails to give access to the further evolution of the system. The relevant time variable to be used in order to reach the second stage of the evolution is the rescaled variable:

$$\tau \doteq \alpha t \tag{57}$$

When the evolution is described in terms of this rescaled time, the first stage collapses into a boundary layer ($\tau \ll 1$) whereas the focus now bears on the second (previously asymptotic) stage. We introduce a rescaled velocity $\tilde{V}(\tau)$, describing a slow motion of the piston:

$$\bar{V}(t) \doteq \alpha \tilde{V}(\tau) \qquad \text{with} \quad \tilde{V}(\tau) = \frac{dX}{d\tau} \tag{58}$$

The motion is now driven by fluctuations and the velocity fluctuations play a crucial role; they allow to define the piston temperature T^{P}:

$$\Delta_2(t) \doteq \alpha \widetilde{\Delta_2}(\tau) \qquad \text{with} \quad \widetilde{\Delta_2} \doteq \frac{k_{\mathrm{B}} T^{\mathrm{P}}}{2m} \tag{59}$$

Finally, mechanical equilibrium has been reached in the first stage of the evolution and only fluctuations around mechanical equilibrium are to be observed in the second stage. We thus introduce:

$$(p^- - p^+)(t) \doteq \alpha \widetilde{\Pi}(\tau) \tag{60}$$

In this second stage, the rescaled quantities are of order 1 and this stage ends when they become of order α. At this point, we should then consider the corresponding equation to order α^2. We now investigate the consequences of the perturbation approach to order 1 in α. Equations (41)–(44) can be written in terms of the rescaled quantities:

$$\begin{cases} \dfrac{\alpha}{\gamma} \dfrac{d\tilde{V}}{d\tau} = \dfrac{\widetilde{\Pi}}{2m} - 2F_1 \tilde{V} + \widetilde{\Delta_2} F_0 \\ \dfrac{\alpha}{\gamma} \dfrac{d\widetilde{\Delta_2}}{d\tau} = -4\widetilde{\Delta_2} F_1 + F_3 \end{cases} \tag{61}$$

Another set of equations describes the evolution of the gas energies (at lowest order in α):

$$\begin{cases} \dfrac{1}{N^-} \dfrac{d\langle E^- \rangle}{d\tau} = -2m \left(\dfrac{A}{N^-} \right) \tilde{V} [F_2^- + \Delta_2 F_0^-] + m \left(\dfrac{A}{N^-} \right) [4\widetilde{\Delta_2} F_1^- - F_3^-] \\ \dfrac{1}{N^+} \dfrac{d\langle E^+ \rangle}{d\tau} = 2m \left(\dfrac{A}{N^+} \right) \tilde{V} [F_2^+ + \Delta_2 F_0^+] - m \left(\dfrac{A}{N^-} \right) [4\widetilde{\Delta_2} F_1^+ - F_3^+] \end{cases} \tag{62}$$

Consistency of the perturbation approach then requires to take the value of all the functions F_j at $\tilde{V}=0$ (as soon as $\tilde{V}$ remains of order $\mathcal{O}(\alpha)$, i.e., $\tilde{V}=\mathcal{O}(1)$), and therefore to replace Eqs. (62) by:

$$\begin{cases} \dfrac{1}{N^-}\dfrac{d\langle E^-\rangle}{d\tau} = -\left(\dfrac{A}{N^-}\right)\tilde{V}p^-_{\text{surf}} + m\left(\dfrac{A}{N^-}\right)[4\widetilde{\Delta_2}F_1^- - F_3^-] \\ \dfrac{1}{N^+}\dfrac{d\langle E^+\rangle}{d\tau} = 2m\left(\dfrac{A}{N^+}\right)\tilde{V}p^+_{\text{surf}} - m\left(\dfrac{A}{N^-}\right)[4\widetilde{\Delta_2}F_1^+ - F_3^+] \end{cases} \tag{63}$$

8. SLAVING PRINCIPLE (CONSISTENCY CONDITION)

Equations (61) show that $\tilde{V}$ and $\widetilde{\Delta_2}$ are slaved to the slow relaxation of the gases towards thermal equilibrium appearing in the $\rho^{\pm}$-dependence of the F_j:

$$\begin{aligned} \frac{\tilde{\Pi}}{2m} &= 2\tilde{V}F_1(0) - \frac{F_3(0)\,F_0(0)}{4F_1(0)} \\ \widetilde{\Delta_2} &= \frac{F_3(0)}{4F_1(0)} \end{aligned} \tag{64}$$

We call *consistency condition* such a lower-order resolution of evolution equations of the general form $\alpha\, dA/d\tau = \text{r.h.s.}(\tau) = \mathcal{O}(1)$, leading to the instantaneous relation $\text{r.h.s.}(\tau)=0$, up to terms of order $\mathcal{O}(\alpha)$. Indeed, it merely follows from a term-wise identification of the expansion in powers of α, given that A actually is a non trivial function of τ when $\alpha \to 0$.[19] It can also be termed a quasi-steady-state approximation (as for instance in the context of enzymatic catalysis[20]). We shall make further encounter of this argument. As mentionned above, the consistency of the lower order perturbation approach also requires to set $V=0$ in functions F_j.

9. AVERAGE ASSUMPTION

At this stage, to simplify our analysis, we introduce the following:

Assumption 2 (Average Assumption). The temperatures and the densities at the surface of piston coincide at order 1 in α with the average energy and density in the fluids in the respectively left/right compartments:

(a) $T^{\pm}_{\text{surf}} = T^{\pm}$ where $N^{\pm}\, k_{\text{B}} T^{\pm} = 2\langle E^{\pm}\rangle$.

(b) $\rho^{\pm}_{\text{surf}} = \rho^{\pm}$ where $\rho^- = \frac{N^-}{AX}$, $\rho^+ = \frac{N^+}{A(L-X)}$, and then $p^{\pm}_{\text{surf}} = p^{\pm} = \rho^{\pm} k_{\text{B}} T^{\pm}$.

From Eq. (45), we get:

$$N^- k_B T^- + N^+ k_B T^+ = 2E_0 - M(\Delta_2 + \bar{V}^2) \tag{65}$$

or equivalently

$$Xp^- + (L-X)\, p^+ = \frac{2E_0}{A} - \frac{M}{A}(\Delta_2 + \bar{V}^2) \tag{66}$$

Therefore:

$$p^- = p_0 + \alpha\left[\left(1-\frac{X}{L}\right)\tilde{\Pi} - \frac{M}{AL}(\widetilde{\Delta_2} + \alpha\tilde{V}^2)\right] \tag{67}$$

$$p^+ = p_0 - \alpha\left[\left(\frac{X}{L}\right)\tilde{\Pi} + \frac{M}{AL}(\widetilde{\Delta_2}) + \alpha\tilde{V}^2)\right] \tag{68}$$

where by definition $p_0 = 2E_0/AL$, and thus:

$$\frac{dp^{\pm}}{d\tau} = \mathcal{O}(\alpha) \tag{69}$$

Moreover, from Eqs. (63) we have:

$$\begin{cases} k_B \dfrac{dT^-}{d\tau} = -2\left(\dfrac{A}{N^-}\right)\tilde{V}p^- + 2m\left(\dfrac{A}{N^-}\right)[4\widetilde{\Delta_2}F_1^- + F_3^-] + \mathcal{O}(\alpha) \\ k_B \dfrac{dT^+}{d\tau} = 2\left(\dfrac{A}{N^+}\right)\tilde{V}p^+ - 2m\left(\dfrac{A}{N^+}\right)[4\widetilde{\Delta_2}F_1^+ + F_3^+] + \mathcal{O}(\alpha) \end{cases} \tag{70}$$

On the other hand, from Assumption 2b (average assumption), we have:

$$p^- = \frac{N^- k_B T^-}{AX} \qquad p^+ = \frac{N^+ k_B T^+}{A(L-X)} \tag{71}$$

which together with Eqs. (70) and (71), yields:

$$\left(\frac{N^-}{A}\right)\frac{1}{p^-}\frac{dp^-}{d\tau} = -3\rho^-\tilde{V} + \frac{2m}{k_B T^-}[4\widetilde{\Delta_2}F_1^- - F_3^-] + \mathcal{O}(\alpha) \tag{72}$$

$$\left(\frac{N^+}{A}\right)\frac{1}{p^+}\frac{dp^+}{d\tau} = 3\rho^+\tilde{V} - \frac{2m}{k_B T^+}[4\widetilde{\Delta_2}F_1^+ - F_3^+] + \mathcal{O}(\alpha) \tag{73}$$

From Eq. (64) follows that:

$$4\widetilde{\Delta_2}F_1^- - F_3^- = 4\widetilde{\Delta_2}F_1^+ - F_3^+ = \frac{1}{F_1}(F_3^- F_1^+ - F_3^+ F_1^-) \tag{74}$$

On the other hand, from Eqs. (67) and (68), we have:

$$\rho^- k_B T^- = \rho^+ k_B T^+ + \mathcal{O}(\alpha) \tag{75}$$

Therefore, since $dp^\pm/d\tau$ is of order α, the consistency condition of the perturbation approach yields from Eqs. (72)–(74) the final relation:

$$\begin{cases} \tilde{V} = \dfrac{m}{3}\left(\dfrac{AL}{E_0}\right)\left(\dfrac{F_3^- F_1^+ - F_3^+ F_1^-}{F_1}\right) + \mathcal{O}(\alpha) \\ \dfrac{\tilde{\Pi}}{2m} = \dfrac{2m}{3}\left(\dfrac{AL}{E_0}\right)(F_3^- F_1^+ - F_3^+ F_1^-) - \dfrac{F_3 F_1}{4F_1} + \mathcal{O}(\alpha) \\ \widetilde{\Delta_2} = \dfrac{F_3}{4F_1} + \mathcal{O}(\alpha) \end{cases} \tag{76}$$

10. DIMENSIONLESS VARIABLES

As usual in perturbation methods, it is convenient to introduce dimensionless variables. From Eq. (14) and Assumption 2a, we have:

$$N^- k_B T^- + N^+ k_B T^+ = 2E_0 - M(\Delta_2 + \bar{V}^2) \tag{77}$$

Let us define:

$$k_B T_0 = \frac{2E_0}{N} = \left(\frac{AL}{N}\right) p_0 \tag{78}$$

It is natural to introduce the dimensionless variable ξ defined by:

$$N^\pm k_B T^\pm \doteq \tfrac{1}{2}(1 \pm 2\xi)[N k_B T_0 - M\alpha(\tilde{\Delta}_2 + \alpha\tilde{V}^2)] \tag{79}$$

We thus have from Eqs. (67) and (68):

$$N^+ k_B T^+ - N^- k_B T^- = 2\xi[ALp_0 - M\alpha(\tilde{\Delta}_2 + \alpha\tilde{V}^2)] \tag{80}$$

$$= p^+ A(L - X) - p^- AX \tag{81}$$

$$= p_0 AL - \alpha AL\left[\frac{X}{L}\tilde{\Pi} + \frac{M}{AL}(\tilde{\Delta}_2 + \alpha\tilde{V}^2)\right] \tag{82}$$

$$-2p_0 AX + \alpha AX\left[\left(\frac{2X}{L} - 1\right)\tilde{\Pi} + \frac{2M}{AL}(\tilde{\Delta}_2 + \alpha\tilde{V}^2)\right] \tag{83}$$

Therefore:

$$2\xi\left[1-\alpha\frac{M}{ALp_0}\tilde{\Delta}_2\right]=\left(1-\frac{2X}{L}\right)+\frac{\alpha}{p_0}\left[\left(\frac{X}{L-1}\right)\frac{M}{AL}\tilde{\Pi}\tilde{\Delta}_2\right]+\mathcal{O}(\alpha^2) \tag{84}$$

The variable ξ remains $\mathcal{O}(1)$ in the second stage of the evolution, hence the consistency of the perturbation approach requires to truncate the previous equation at lower order in α, which yields:

$$\xi=\frac{1}{2}-\frac{X}{L}+\mathcal{O}(\alpha) \tag{85}$$

According to this the consistency condition, we have:

$$\frac{d\xi}{d\tau}=-\frac{\tilde{V}}{L} \tag{86}$$

and from Eq. (76), we obtain our final equation:

$$\frac{d\xi}{d\tau}=-\frac{m}{3}\left(\frac{A}{E_0}\right)\frac{1}{F_1}\left(F_3^-F_1^+-F_3^+F_1^-\right) \tag{87}$$

11. MAXWELLIAN IDENTITIES

Let us introduce at this point the following:

Assumption 3 (Maxwellian Identities)

(a) The relation between the functionals F_1 and F_3 obtained when $\rho^\pm$ are Maxwellian remains valid here at order α, i.e.,

$$F_3^\pm(V)=\frac{2k_BT^\pm}{m}F_1^\pm(V)-VF_2^\pm(V)+\mathcal{O}(\alpha)\qquad\text{provided}\quad V=\mathcal{O}(\alpha) \tag{88}$$

(b) $F_1^\pm=F_1^\pm(V=0)$ coincide at order α with the value given by Maxwellian distributions, i.e.,

$$F_1^\pm(0)=\mp\rho^\pm\sqrt{\frac{k_BT^\pm}{2m\pi}}+\mathcal{O}(\alpha) \tag{89}$$

From Assumption 3a, we have:

$$F_3^-F_1^+-F_3^+F_1^-=\frac{2k_B}{m}\left(T^--T^+\right)F_1^-F_1^+ \tag{90}$$

and thus:

$$\frac{d\xi}{d\tau} = -k_B(T^- - T^+)\left(\frac{2A}{3E_0}\right)\frac{F_1^- F_1^+}{F_1} \tag{91}$$

$$\tilde{V} = k_B(T^- - T^+)\left(\frac{2AL}{3E_0}\right)\frac{F_1^- F_1^+}{F_1} \tag{92}$$

$$\tilde{\Delta}_2 = \frac{k_B}{2m}\frac{(T^- F_1^- - T^+ F_1^+)}{F_1} \tag{93}$$

$$\frac{\tilde{\Pi}}{2m} = k_B(T^- - T^+)\left[\left(\frac{4AL}{3E_0}\right)F_1^- F_1^+ + \left(\frac{E_0}{AL}\right)\frac{\tilde{\Delta}_2}{k_B^2 T^- T^+}\right] \tag{94}$$

We remark that whatever are the distributions $\rho^{\pm}(v, t)$, then as long as Assumption 3a is satisfied, we have $\tilde{V} > 0$ if and only if $T^+ > T^-$, i.e., the piston moves in the direction of the warmer side, which is typical of heat transfer since $p^+ = p^- + \mathcal{O}(\alpha)$.

To obtain an explicit equation for the evolution, we shall now use the Assumption 3b above, together with the fact that $\rho^{\pm} k_B T^{\pm} = p_0 + \mathcal{O}(\alpha)$. We thus have:

$$\frac{F_1^- F_1^+}{F_1} = \sqrt{\frac{2}{m\pi}}\left(\frac{E_0}{AL}\right)\frac{1}{\sqrt{k_B}}\frac{1}{\sqrt{T^-} + \sqrt{T^+}} \tag{95}$$

and Eq. (87) leads to the main result of our paper:

$$\frac{d\xi}{d\tau} = -\frac{1}{3}\frac{1}{L}\sqrt{\frac{E_0}{N}}\sqrt{\frac{8}{m\pi}}\left[\sqrt{\frac{N}{N^+}(1+2\xi)} - \sqrt{\frac{N}{N^-}(1-2\xi)}\right] \tag{96}$$

Finally introducing the dimensionless variable s by:

$$s = \tau\frac{2}{3L}\sqrt{\frac{k_B}{m\pi}}\sqrt{\frac{2N^-}{N}T_0^- + \frac{2N^+}{N}T_0^+} \tag{97}$$

we have:

Main Result. Under Assumptions 1–3, the evolution of the system in the second stage is described in terms of the dimensionless variables by

$$\frac{d\xi}{ds} = -\left[\sqrt{\frac{N}{2N^+}(1+2\xi)} - \sqrt{\frac{N}{2N^-}(1-2\xi)}\right] \tag{98}$$

together with the "initial condition" (in fact the matching condition of the boundary-layer-type perturbation approach) which is from Eq. (85):

$$\xi(s=0) = \frac{1}{2} - \frac{X_{\mathrm{ad}}}{L} \tag{99}$$

with X_{ad} being the position of the piston at the end of the first stage, Eq. (55), i.e., at mechanical equilibrium. Integrating Eq. (96) with Eq. (99) yields the evolution $\xi = \xi(s)$ from which we obtain the position and the temperatures:

$$X = L\left(\frac{1}{2} - \xi\right) \tag{100}$$

$$T^{\pm} = (1 \pm 2\xi)\,\frac{N^{-}T_0^{-} + N^{+}T_0^{+}}{2N^{\pm}} \tag{101}$$

together with $\widetilde{V}$, $\widetilde{\Delta_2}$, and $\widetilde{\Pi}$ obtained from Eqs. (92)–(94) and Eq. (95), i.e.,

$$\begin{cases} \widetilde{V} = \dfrac{1}{3}\sqrt{\dfrac{8k_{\mathrm{B}}}{\pi m}}\,(\sqrt{T^{+}} - \sqrt{T^{-}}) \\ \widetilde{\Delta_2} = \dfrac{k_{\mathrm{B}}}{2m}\sqrt{T^{-}T^{+}}, \qquad \text{i.e.,} \quad T^{\mathrm{P}} = \sqrt{T^{+}T^{-}} \\ \widetilde{\Pi} = \left(\dfrac{E_0}{AL}\right)\left(\dfrac{T^{+} - T^{-}}{\sqrt{T^{-}T^{+}}}\right)\left(\dfrac{16}{3\pi} - 1\right) \end{cases} \tag{102}$$

From Eq. (96), the second stage of the evolution described here will proceed until:

$$\xi = \xi_f = \frac{N^{+} - N^{-}}{2N} + \mathcal{O}(\alpha) \tag{103}$$

which implies:

$$X = X_f = \frac{N^{-}}{N} + \mathcal{O}(\alpha) \tag{104}$$

$$T^{\pm} = T_f^{\pm} = \frac{2E_0}{Nk_{\mathrm{B}}} + \mathcal{O}(\alpha) \tag{105}$$

$$\tilde{\Delta}_2 = \frac{k_{\mathrm{B}}T_f^{\mathrm{P}}}{2m} = \frac{E_0}{mN} + \mathcal{O}(\alpha), \qquad \text{i.e.,} \quad T_f^{\mathrm{P}} = T_f^{\pm} + \mathcal{O}(\alpha) \tag{106}$$

$$\widetilde{V} = \mathcal{O}(\alpha) \tag{107}$$

$$\widetilde{\Pi} = \mathcal{O}(\alpha) \tag{108}$$

At this point, corrections in α^2 have to be included; the relevant perturbation approach to get the further evolution would use the rescaled time $\tau_2 \doteq \alpha^2 t$ and consider also Δ_3 and Δ_4, and so on for the successive stages of the relaxation towards complete equilibrium (Maxwellian distributions for the gas particles).

12. REMARKS

(1) Our main result Eq. (98) gives an equation totally independent of the parameters of the problem, except for the ratio $2N^{\pm}/N$, which will be 1 in our simulations.

(2) Expression (102) for V and Eq. (96) describing the evolution, coincide with the equations obtained in ref. 9 using qualitative arguments.

(3) Equation (98) for $\xi(s)$ was also derived by J. L. Lebowitz[22] using either qualitative arguments or the fact that the evolution of the piston will be described by an Ornstein–Uhlenbeck process.

(4) The consistency between the short time solution (obtained in ref. 10 with $\alpha = 0$) and the solution for large time, here obtained by considering evolution equations with respect to rescaled variable τ and restricted at first order in α, amounts to match the infinite-time limit $t \to \infty$ of the zero-order solution with the initial condition in $\tau = 0$ of the rescaled perturbation large-time solution, which makes sense only for large R.

(5) From Eq. (102), we see that $\tilde{\Pi} > 0$ if and only if $T^+ > T^-$. As we have seen above, it is precisely the condition for $\tilde{V} > 0$. Therefore work is delivered to the warmer side but more heat is extracted from this warmer side so that its energy decreases.

(6) We should compare the expression obtained for the velocity Eq. (102):

$$V = \frac{2}{3M}\sqrt{\frac{8k_B m}{\pi}}\left(\sqrt{T^+} - \sqrt{T^-}\right) \tag{109}$$

with the expression obtained in ref. 7 for the stationnary state of the piston in an infinite cylinder under the condition that the pressures on both sides are equal:

$$V_{st} = \frac{1}{M}\sqrt{\frac{\pi k_B m}{8}}\left(\sqrt{T^+} - \sqrt{T^-}\right) \tag{110}$$

The velocity V is larger than V_{st} by a factor $16/3\pi$. It is related to the fact that in ref. 7 we could impose the condition that the pressures are the same,

while in the present situation (cylinder with finite length) the pressures are different, determined by the equations for the evolution which gives:

$$p^- - p^+ = \frac{2m\tilde{I}\tilde{I}}{M} + \mathcal{O}(\alpha^2) \geqslant 0 \tag{111}$$

13. NUMERICAL SIMULATIONS

In order to check Assumptions 1–3 and to test our main results, in particular Eqs. (96)–(108), we have conducted a large number of simulations. We have considered a one-dimensional system of fixed length L. The mass of the fluid particles is $m = 1$ and the mass M of the piston is varied in the different simulations. Initially the piston is at the position X_0 with velocity $V_0 = 0$ and a configuration of N^- and N^+ particles is taken at random from Maxwellian distributions with temperatures T_0^- and T_0^+. The system is then led to evolve according to the law of purely elastic collisions Eq. (1). The position X of the piston is recorded, together with the temperatures and pressures in the left and right compartments defined by the average (kinetic) energy and the equation of state $p^\pm = \rho^\pm k_B T^\pm$.

To conduct these numerical simulations, one is confronted with the following dilemma. In order to reduce the fluctuations to a minimum to obtain significant results, one should take $N^\pm$ very large. This implies that M should also be very large in order to have a strong damping. However since the second stage proceeds on a time scale $\tau = 2t/M$, this implies that one has to follow an enormous number of collisions, which needs a very long computer time. For this reason, to study the first stage of the evolution, we have considered up to 2 millions particles, but we were forced to reduce drastically this number of particles to investigate the second stage, i.e., to approach thermal equilibrium. In all simulations presented below we have taken:

$$k_B = 1, \quad m = 1, \quad L = 60, \quad N^- = N^+ = \bar{N} = \frac{N}{2}$$

$$X_0 = 10, \quad V_0 = 0, \quad T_0^- = 1, \quad T_0^+ = 10, \quad \text{i.e., } \rho_0^- = 5\rho_0^+ \text{ and } p_0^- = \frac{p_0^+}{2}$$

For these initial conditions, the end of the first stage, i.e., mechanical equilibrium, is characterized by Eqs. (51)–(55) which give:

$$X_{ad} = 8.42, \quad T_{ad}^- = 1.54, \quad T_{ad}^+ = 9.46, \quad p_{ad}^- = p_{ad}^+ = 0.1833\bar{N} \tag{112}$$

We concluded in ref. 10 that in the first stage, the motion is deterministic and depends strongly on $R = \bar{N}/M$ if $R < 1$, but become independent of R if $R > 4$. In particular for $R < 1$, it is weakly damped and the period of oscillations we have computed coincide with the period obtained from simulations and from hydrodynamics assuming adiabatic oscillations. On the other hand for $R > 10$, the motion is strongly damped and is independent of $N^{\pm}$ and M as soon as M is large enough. Moreover at the end of the first stage, after mechanical equilibrium has been reached, simulations shows that oscillations start to appear. These oscillations were interpreted as a consequence of the fact that the velocity distributions in the fluid were not Maxwellian.

In Fig. 1, we have plotted the first part of the evolution, i.e., the adiabatic approach towards mechanical equilibrium, in function of time t for $N^{\pm} = 2 \cdot 10^4$, $M = 10^5$ (i.e., $R = 0.2$, weak damping) and for $N^{\pm} = 3 \cdot 10^5$, $M = 3 \cdot 10^4$ (i.e., $R = 10$, strong damping). From the simulations the values obtained at mechanical equilibrium are:

$$\begin{aligned} X_{\text{obs}} &= 8.33 \pm 0.05, \qquad T^{-}_{\text{obs}} = 1.52 \pm 0.04, \\ T^{+}_{\text{obs}} &= 9.48 \pm 0.04, \qquad p^{-}_{\text{obs}} = p^{+}_{\text{ad}} = 0.183\bar{N} \end{aligned} \tag{113}$$

in good agreement with the analytical values Eq. (112). Other graphs for this adiabatic evolution can be found in ref. 10.

In Figs. 2–8, we have plotted the second part of the evolution, i.e., the approach towards thermal equilibrium in terms of the scaled variable $\tau = 2t/M$, obtained from our equations and from simulations with $N^{\pm} = 3 \cdot 10^4$ and $M = 100, 200, 1000$. The position in function of τ is presented in Fig. 2, where no significant difference can be seen for the different values of M. Figure 3 shows a zoom around $\tau = 0$ and $\tau = 16$ for $M = 200$

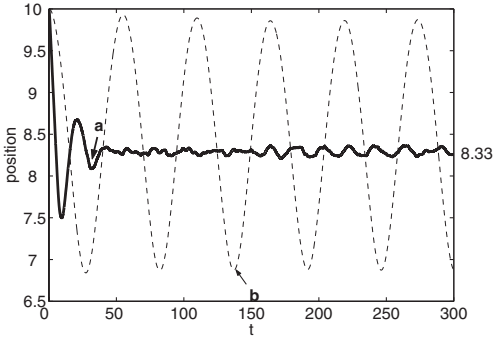

Fig. 1. Adiabatic stage of the evolution for $T_0^{-} = 1$ and $T_0^{+} = 10$. (a) Strong damping: $N^{\pm} = 3 \cdot 10^5$, $M = 3 \cdot 10^4$, $R = 10$. (b) Weak damping: $N^{\pm} = 2.10^4$, $M = 10^5$, $R = 0.2$.

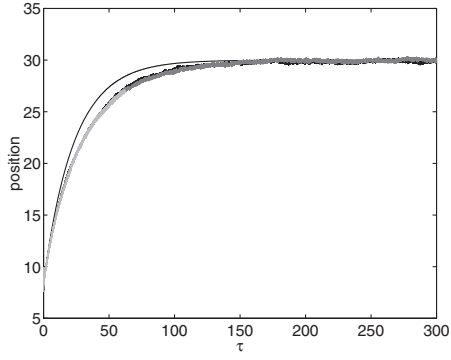

Fig. 2. Approach to thermal equilibrium for $N^{\pm} = 3 \cdot 10^4$ and $M = 100, 200, 1000$ compared with the solution of Eq. (96) (the real time is $t = \tau M/2$).

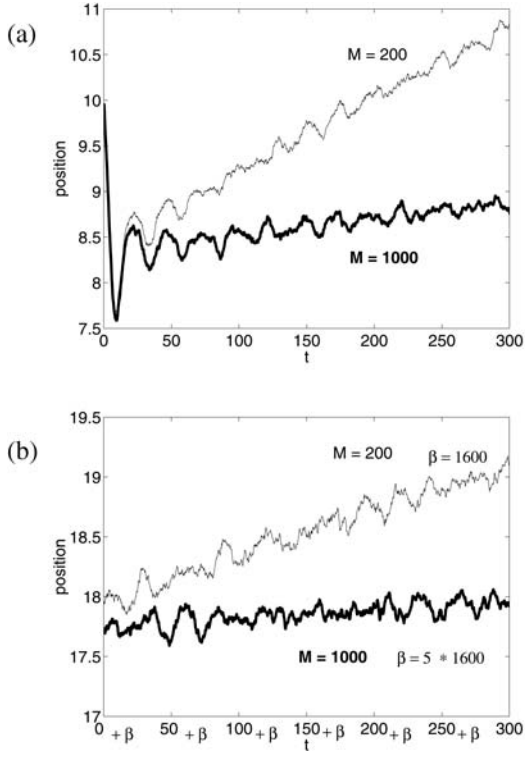

Fig. 3. Zoom on the evolution of Fig. 2 in function of the real time t for $M = 200$ (light curve) and $M = 1000$ (bold curve) around $\tau = 0$ and $\tau = 16$.

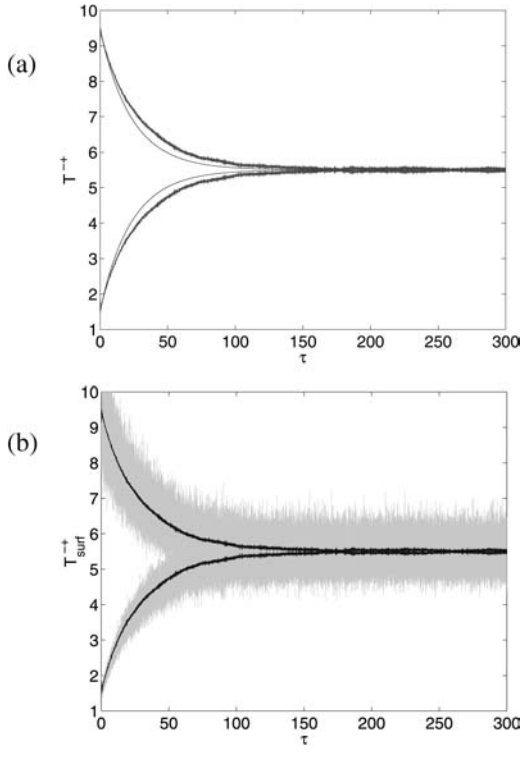

Fig. 4. (a) Evolution of the temperatures $T^{\pm}(\tau)$ for $N^{\pm} = 3 \cdot 10^4$ and $M = 200$ compared with Eqs. (96) and (101); (b) Surface temperatures $T^{\pm}_{surf}(\tau)$ compared with bulk temperatures $T^{\pm}(\tau)$.

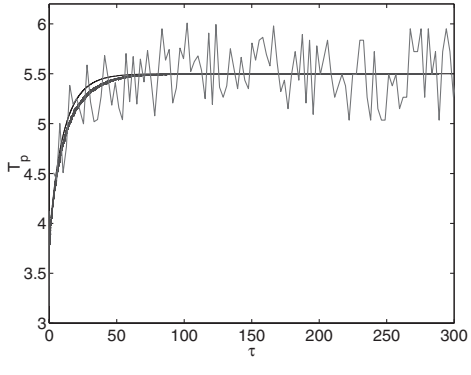

Fig. 5. Temperature of the piston ($N^{\pm} = 3 \cdot 10^4$, $M = 200$): (a) from $T^{P} = \sqrt{T^{+}T^{-}}$ (Eq. (102) with $T^{\pm}$ obtained from Eq. (96) (light curve); (b) from $T^{P} = \sqrt{T^{+}T^{-}}$ with $T^{\pm}$ obtained from the simulations (dark curve); (c) from the definition Eq. (77) with $k_B T^{P} = M\Delta_2$ (stochastic curve).

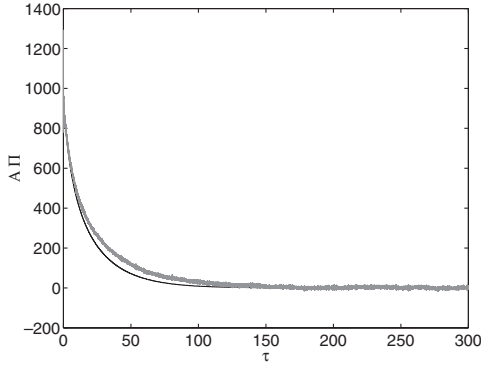

Fig. 6. Scaled pressure difference $A\tilde{\pi} = A(p^- - p^+)\, M/2$ computed from Eq. (102) with $T^\pm$ obtained from Eq. (96) and from simulations (with $N^\pm = 3 \cdot 10^4$, $M = 200$).

and $M = 1000$, expressed in terms of the real time $t = \tau M/2$. It shows that on the time interval $(t, t+300)$ considered in this zoom, the evolution (fast local relaxation slaved to the slow evolution of the system) will tend to be independent of M for M sufficiently large (fluctuations around mechanical equilibrium) while on the large scale the evolution is scaled by $1/M$, i.e., appears independent of M when expressed in function of $\tau = 2t/M$. In Fig. 4, we present the temperatures $T^\pm$ together with the surface temperatures $T^\pm_{\text{surf}}$ computed from the incoming particles at distance 1 from the piston. In Fig. 5 the temperature of the piston is plotted using either Eq. (102), i.e., $T^{\text{P}} = \sqrt{T^+ T^-}$ or the definition (77) with $k_{\text{B}} T^{\text{P}} = M \Delta_2$; one should note that using the definition (77) implies a factor which is in our

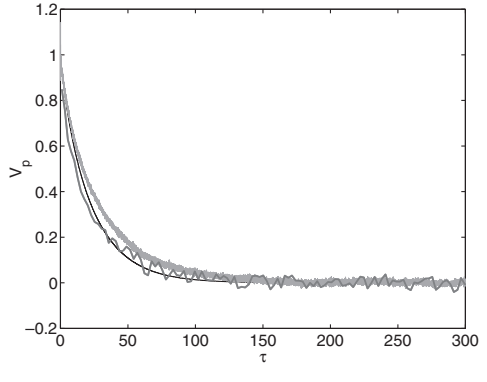

Fig. 7. Scaled velocities ($N^\pm = 3 \cdot 10^4$, $M = 200$): (a) from Eq. (96) (light curve); (b) from Eq. (102) with $T^\pm$ obtained from the simulations (dark curves); (c) average velocity obtained from the simulations.

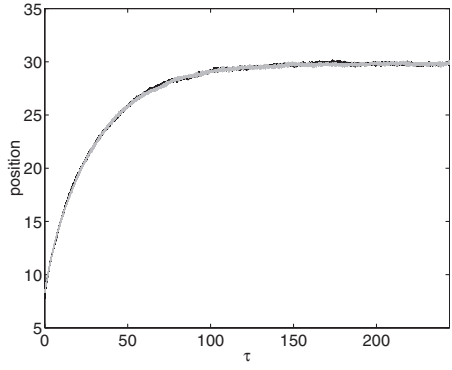

Fig. 8. Evolution for $N^{\pm} = 3 \cdot 10^4$, $M = 200$ compared with the evolution starting from initial conditions Eq. (113) and Maxwellian velocity distributions.

simulations 30000 $(T^+ + T^- - 11)$ and thus the fluctuations are very large. The other scaled quantities of order 1, i.e., $\tilde{\Pi} = M(p^- - p^+)/2$ and $\tilde{V} = MV/2$ (recall that $m = 1$) obtained from the simulations and our Eqs. (102) are represented in Figs. 6 and 7. Finally, since the distributions of velocities of the gases do not remain Maxwellian, we compare in Fig. 8 the previous evolution with the evolution starting from the (adiabatic) initial conditions (113) with Maxwellian distributions. We observe no significant difference.

14. CONCLUSIONS

Using a two-time-scale perturbation approach, we have shown that the evolution of a piston with large but finite mass proceeds in two stages. For $R > 10$, there is strong damping and in the first stage, at times $t = \mathcal{O}(1)$, the evolution is similar to the one previously discussed in the thermodynamic limit:[10] it is a deterministic, adiabatic evolution towards mechanical equilibrium. At the end of this first stage, adiabatic oscillations appear which are interpreted as associated with the fact that the velocity distributions of the gas particles are not Maxwellian. In particular, simulations show that in the adiabatic evolution:

$$\frac{1}{T^-}\frac{dT^-}{dt} \approx -1.8\,\frac{1}{X}\frac{dX}{dt} \tag{114}$$

In the second stage, at times $t = \mathcal{O}(M)$, a fluctuation-driven regime develops, leading to a relaxation towards thermal equilibrium. In this regime, the motion of the piston is slaved (i.e., it adapts on a time scale

$\mathcal{O}(1)$) to the slowly relaxing (on a time scale $\mathcal{O}(M)$) asymmetry of the thermal fluctuations on each side. Moreover, since the motion of the piston is now stochastic, one has to introduce the temperature of the piston. Therefore the fluctuations in the piston motion produce a heat transfer from the warmer to the colder side which is larger than the work produced by the piston motion. In particular, in this stage, simulations as well as equations show that now:

$$\frac{1}{T^-}\frac{dT^-}{dt} \approx \frac{1}{X}\frac{dX}{dt} \tag{115}$$

In this regime, the piston is no longer adiabatic but the time involved to reach thermal equilibrium is of order M.

In the first stage, simulations are in very good agreement with the predicted values for the position of the piston and the temperatures of the gases at mechanical equilibrium. Moreover for weak damping ($R < 1$) the observed values for the period coincide within a few per cent with the predicted values from our equations, as well as with the values obtained from thermodynamics assuming "adiabatic oscillations." However the observed values for the damping coefficient disagree strongly with the computed values. Unravelling the origin of the friction, we must conclude that there are two different friction coefficients. A first coefficient describes the stabilization of the piston velocity in the infinite length situation, or for the case of finite length (and $R > 10$) until the first recollision on the piston takes place. It is related to the shock waves propagating uniformly on both sides of the surface of the piston. The second coefficient describing the damping of the oscillations is associated with the rebouncing of these shock waves on the surface of the cylinder and of the piston. It appears to be related to the relative velocity of the shock waves with respect to the piston velocity.[11]

For the second stage, as can be seen on Figs. 2 to 7, our main conclusions are all verified by numerical simulations with high degree of accuracy. One should however observe that the agreement would be "perfect" if we would consider, instead of Eq. (98), the equation:

$$\frac{d\xi}{ds} = -0.8\left[\sqrt{\frac{N}{2N^+}(1+2\xi)} - \sqrt{\frac{N}{2N^-}(1-2\xi)}\right] \tag{116}$$

The origin of the discrepancy could be traced to our Assumption 3b where we use the Maxwellian values for $F_1^{\pm}(V=0)$. It is known[9] that at this stage the distribution functions $f^{\pm}(v)$ differ considerably from the Maxwellian distributions around $v = 0$. Of course this discrepancy could

also come from the average assumption (Assumption 2); however from Fig. 4, we see that the surface temperatures oscillate around the bulk temperatures (on a time scale t) and thus we expect this technical assumption to have no important consequences.

Let us remark that if we compute the time auto-correlation function $C(t)$ of the position X of the piston and the associated power spectrum $S(\omega)$ (the Fourier transform of $C(t)$), then from the simulation we could expect a three-peak structure as observed in the equilibrium state by White *et al.*[16] recovering a standard result in hydrodynamics.[23] The central peak describes (according to the fluctuation-dissipation theorem) both

– the (slow) relaxation towards thermal equilibrium, described by a term $\sim e^{-t/\tau_{th}}$ in the time auto-correlation function $C(t)$, with $\tau_{th} \sim 1/\alpha \sim M$; the width of the peak is $\mathcal{O}(1/M)$.

– and the thermal fluctuations of the piston as described by $\Delta_2(\infty)$.

The two lateral peaks, slowly evolving in time with $X(\tau)$ and $T^{\pm}(\tau)$ correspond to the fast adiabatic and deterministic relaxation towards the instantaneous mechanical equilibrium, slaved to the values of $X(\tau)$ and $T^{\pm}(\tau)$. It also gives (fluctuation-dissipation theorem in this quasi-equilibrium state) a contribution to the fluctuations of the piston motion. The width of these peaks is independent of M.

The perturbation approach can be carried on at higher orders; it would give access to the following stages of the relaxation, at increasing time scales $\mathcal{O}(1/\alpha^n) \sim \mathcal{O}(M^n)$, involving higher moments Δ_{2n-1} and Δ_{2n}. Maxwellian distributions for the velocity distributions inside the gases would be reached only in the final stages of this relaxation (namely at times $t \sim \mathcal{O}(M^n)$ with n large). However the factorization property will be violated at these orders and we can not say what would be the consequences of this violation on the results obtained within our Boltzmann's equation approach.

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